

Flow of wet granular materials: a numerical study

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We simulate dense assemblies of frictional spherical grains in steady shear flow under controlled normal stress P in the presence of a small amount of an interstitial liquid, which gives rise to capillary menisci, assumed isolated (pendular regime), and to attractive forces, which are hysteric: menisci form at contact, but do not break until grains are separated by a finite rupture distance. The system behavior depends on two dimensionless control parameters: inertial number I and reduced pressure $P^* = aP/(\pi\Gamma)$, comparing confining forces $\sim a^2P$ to meniscus tensile strength $F_0 = \pi\Gamma a$, for grains of diameter a joined by menisci with surface tension Γ . We pay special attention to the quasi-static limit of slow flow and observe systematic, enduring strain localization in some of the cohesion-dominated ($P^* \sim 0.1$) systems. Homogeneous steady flows are characterized by the dependence of internal friction coefficient μ^* and solid fraction Φ on I and P^* . We also record normal stress differences, fairly small but not negligible, and increasing for decreasing P^* . The system rheology is moderately sensitive to saturation within the pendular regime, but would be different in the absence of capillary hysteresis. Capillary forces have a significant effect on the macroscopic behavior of the system, up to P^* values of several units, especially for longer force ranges associated with larger menisci. The concept of effective pressure may be used to predict an order of magnitude for the strong increase of μ^* as P^* decreases but such a crude approach is unable to account for the complex structural changes induced by capillary cohesion, with a significant decrease of Φ and different agglomeration states and anisotropic fabric. Likewise, the Mohr-Coulomb criterion for pressure-dependent critical states is, at best, an approximation valid within a restricted range of pressures, with $P^* \geq 1$. At small enough P^* , large clusters of interacting grains form in slow flows, in which liquid bonds survive shear strains of several units. This affects the anisotropies associated to different interactions, and the shape of function $\mu^*(I)$, which departs more slowly from its quasistatic limit than in cohesionless systems (possibly explaining the shear banding tendency).

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I. INTRODUCTION

Over the last decade, constitutive modelling of dense granular flows has been proposed [1, 2] in terms of internal friction laws directly applying to normal stress-controlled steady shear flows, for which the internal state of the material is characterized by a single dimensionless number, the inertial parameter I [3]. Number I might be regarded as a reduced, dimensionless form of shear rate $\dot{\gamma} = \partial v_1 / \partial x_2$, related to the stress σ_{22} normal to flow direction x as $I = \dot{\gamma} \sqrt{\frac{m}{a\sigma_{22}}}$, m denoting the particle mass and a its diameter. The constitutive law relating the effective internal friction coefficient, μ^* , defined as a stress ratio, $\mu^* = \sigma_{12}/\sigma_{22}$, to inertial number I should be supplemented with a similar relation of solid fraction Φ to I [1, 4–6]. I characterizes dynamical effects, and the quasistatic limit is that of vanishing I . In this limit of $I \rightarrow 0$, the material is in the so-called critical state

of soil mechanics [7], i.e., quasistatic plastic shear flow at constant solid fraction Φ_c , under constant stresses and effective internal friction μ_c^* . In various experimental and numerical studies, the constitutive law, suitably generalized, was shown to apply to different grain shapes and flow geometries [8–10]. A major advantage of the “ $\mu^*(I)$ and $\Phi(I)$ ” approach is its ability to deal with both the quasistatic limit and the rigid limit without any divergence or singularity. On regarding inertial number I as the sole state parameter in a granular material in shear flow, it is implicitly assumed that small contact deflections due to the finite elastic stiffness of the grains are irrelevant – this is the rigid limit.

In the presence of attractive forces between neighboring grains, contacts are endowed with a finite tensile strength F_0 , whence a new dimensionless parameter, P^* , a reduced pressure comparing the applied confining stress P (say, the controlled normal stress value σ_{22} in shear flow) to force scale F_0 , as $P^* = \frac{a^2\sigma_{22}}{F_0}$ (similarly a “cohesion number” $\eta = 1/P^*$ was defined in Ref. [11]). Under small P^* , cohesion stabilizes loose structures [12, 13], which collapse upon increasing P^* [14, 15]. In steady shear flow, generalization of rheological laws to the cohesive case involve internal friction coefficient and density, functions of both numbers I and P^* or η [16].

In wet granular materials cohesion arises from capillary

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forces due to small liquid bridges joining particles touching or in close vicinity to each other [17, 18]. The effect of such forces has been investigated in quasistatic deformation [19–21], and some of its consequences in terms of microstructure were discussed [22]. In the *pendular regime* of saturation [17, 18] those bridges are small enough and do not merge, so that capillary forces are pairwise additive. Those attractive forces act as a source of cohesion, and are also characterized by a small range and some dependence on intergranular distance, as a liquid meniscus might join grains that are not in contact.

A traditional approach of partially saturated granular materials in geomechanics [23], which has been investigated in recent DEM studies [24, 25], is to resort to the concept of *effective stresses*, or stresses such that, if applied to the dry material, would produce the same deformation and plastic flow of the granular skeleton as in the ones observed in the wet material. Proposed definitions of such an effective stress tensor in the unsaturated case generalize the Terzaghi principle [26] applying to saturated media, and involve a correction of the average pressure related to saturation and capillary pressure [27].

On the macroscopic scale, the effect of adhesive forces are sometimes described in the quasistatic limit of slow flow by the phenomenological Mohr-Coulomb law [7, 28, 29],

$$\sigma_{12} = c + \mu_1^* \sigma_{22}, \quad (1)$$

characterized by macroscopic cohesion c and internal friction coefficient μ_1^* .

The present paper investigates the constitutive laws applying to wet model granular materials, in the pendular regime, and discusses the influence of capillary effects on macroscopic behavior and microstructure. Similarly to refs. [11, 16], the rheology and micromechanical aspects are studied for varying P^* and I values (with special emphasis on the quasistatic limit of $I \rightarrow 0$). As in dry granular systems and in previous studies on 2D cohesive materials the material rheology is described in terms of apparent friction coefficient (stress ratio) μ^* and solid fraction Φ as functions of I and P^* , and the applicability of a Mohr-Coulomb relation is tested. Rheological and microstructure features as normal stress differences and formation of large clusters bonded by liquid bridges are also investigated.

In the following, we first introduce (Sec. II) the microscopic ingredients of the model material, and report then, in Sec. III, on the conditions in which homogeneous steady states are observed in shear flows, enabling material constitutive laws to be deduced. Such laws are measured, depending on the relevant dimensionless parameters and on some features of the microscopic model, in Sec. IV. Next, in Sec. V, we investigate the role of capillary forces and distant interactions in the material rheology, and revisit the traditional concepts of effective stress and Mohr-Coulomb cohesion. Additional studies of microstructural and micromechanical aspects follow: force distributions (Sec. VI), agglomeration effects

(Sec. VII), structural anisotropy (Sec. VIII). The results are discussed and put in perspective in the final, conclusive section IX.

II. MODEL MATERIAL AND SIMULATION SETUP

We consider a granular assembly composed of N equal-sized spherical beads of diameter a , made of a material with Young modulus E and Poisson ratio ν . The contacts are frictional, satisfying Coulomb's law with friction coefficient μ . The granular flow is set by imposing a uniform shear rate $\dot{\gamma}$ to a rectangular parallelepipedic cell with edge lengths $(L_\alpha)_{1 \leq \alpha < 3}$. In order to avoid wall effects periodic boundary conditions are used in all three directions. The periodicity, in the direction of the flow gradient (direction 2), is applied with the Lees-Edwards procedure [30] and in the two other directions the boundary condition is simple periodic. The system size L_2 is allowed to fluctuate in order to keep the normal stress σ_{22} constant equal to a prescribed value P while L_1 and L_2 are fixed [5].

A. Force model

Elastic and frictional forces are jointly implemented in contacts as in Ref. [31], in which a simplified Hertz-Mindlin-Deresiewicz force model is used for the elastoplastic contact behaviour. This model combines the normal Hertz force F_N , depending on contact deflection h as

$$F_N = \frac{\tilde{E}\sqrt{a}}{3} h^{3/2}, \quad (2)$$

in which we introduced notation $\tilde{E} = E/(1 - \nu^2)$, with a tangential elastic force \mathbf{F}_T , to be evaluated incrementally in each time step of the simulation. The simplification of tangential elasticity adopted is that of Ref. [31], involving a constant ratio $(2 - 2\nu)/(2 - \nu)$ of tangential (K_T) to normal (K_N) stiffnesses in contacts, both depending on F_N , as, from (2), one has $K_N = \frac{dF_N}{dh} \propto F_N^{1/3}$. Caution should be exercised to avoid spurious creation of elastic energy with such laws, and K_T should be suitably rescaled in cases of decreasing normal force and deflection. For the details of the elastic model, for the enforcement of the Coulomb condition $\|\mathbf{F}_T\| \leq \mu F_N$, and for the objective implementation of the force law, with due account of all possible motions of a pair of contacting grains, the reader is referred to [31].

An estimate of the typical contact deflection under confining stress P defines a dimensionless parameter, stiffness number κ [32], such that $h/a \propto \kappa^{-1}$. For a Hertzian contact, one may use [31]

$$\kappa = \left(\frac{\tilde{E}}{P}\right)^{2/3}. \quad (3)$$

Two values of κ , 8400 and 39000, used in this study, respectively correspond to glass beads with $E = 70$ GPa and $\nu = 0.3$ under pressures 100 kPa and 10 kPa, as in Ref. [5]. Finally, the force model of [31] which is used here may also comprise a viscous damping term opposing normal relative motion of contacting grains, chosen to correspond to a restitution coefficient close to zero in normal collisions. We do not comment this feature, as it was shown [1, 5] to have very little influence in the slow shear flows of the present study.

The presence of a small amount of an interstitial wetting liquid introduces additional capillary forces, transmitted between contacting or neighboring grains by a liquid bridge, or meniscus, as sketched in Fig. 1. We

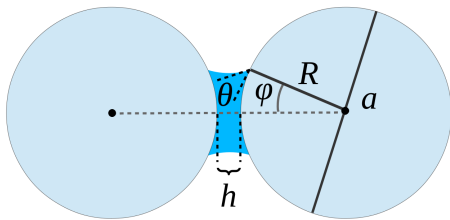


FIG. 1: (Color online) A meniscus between two spherical grains of diameter $a = 2R$, with distance h between solid surfaces, filling angle φ , contact angle θ .

consider a perfectly wetting liquid, with contact angle θ equal to zero. As in Ref. [17], we assume that the menisci only form for touching particles, but breaks for gaps larger than a certain rupture distance D_0 , as observed in [33]. D_0 relates to meniscus volume V as $D_0 \simeq V^{1/3}$ [34–37].

For the attractive force between particles separated by distance $h \leq D_0$ we adopt the Maugis approximation [38], which is appropriate for small enough meniscus volume, for its simplicity. The maximum attractive force (tensile strength) is reached for contacting particles, and equal, according to this model, to $F_0 = \pi a \Gamma$ (Γ is the liquid surface tension) independently of the meniscus volume. The capillary force varies with the distance h between particle surfaces as

$$F^{\text{Cap}} = \begin{cases} -F_0 & h \leq 0 \\ -F_0 \left[1 - \frac{1}{\sqrt{1 + \frac{2V}{\pi a h^2}}} \right] & 0 < h \leq D_0 \\ 0 & h > D_0 \end{cases} \quad (4)$$

$h < 0$ corresponds to an elastic deflection of the particles in contact. This formula is a simpler, analytical form of the toroidal approximation with the “gorge method” [34] for the capillary force in a meniscus, which describes the meniscus as limited by circular arcs in a plane containing the two sphere centers. An alternative form was given by Willett *et al.* [35], while Soulié *et al.* [20, 39, 40] proposed a parametrized numerical solution. Fig. 2 compares functions $F^{\text{Cap}}(h)$ according to Maugis and to Soulié *et al.*. Some comparisons between several formulae and experiments are given in [36].

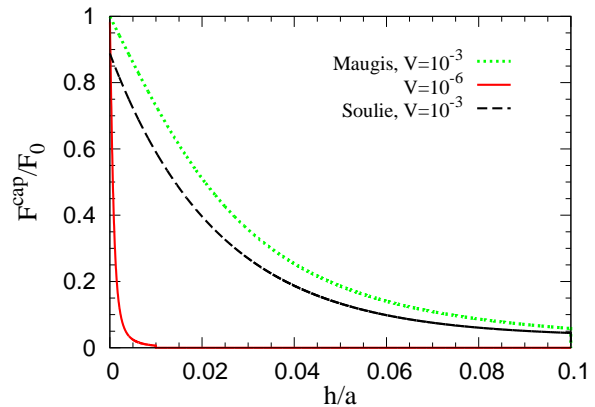


FIG. 2: (Color online) Force law $F^{\text{Cap}}(h)$, for two different meniscus volumes, according to the Maugis model and to the Soulié formula.

B. Saturation range of pendular state

The morphology of partially saturated granular materials depends on the liquid content [18, 41]. The present study, like a number of previous ones [19, 21, 40], is restricted to the *pendular state* of low saturations, in which the wetting liquid is confined in bonds or menisci joining contacting grains. Liquid saturation S is defined as the ratio of liquid volume Ω_l to interstitial volume Ω_v . It is related to meniscus volume V , solid fraction $\Phi = 1 - \Omega_v/\Omega$ and wet coordination number z (the average number of liquid bonds on one grain) as:

$$S = \frac{\Omega_l}{\Omega_v} = \frac{3z}{\pi} \frac{\Phi}{1 - \Phi} \frac{V}{a^3}. \quad (5)$$

In our study, we fix the value of meniscus volume V . Such a choice does not conserve the total liquid volume, which is proportional to the varying coordination number z of liquid bonds. Its consequences have to be assessed, and we shall check that the results are not significantly affected, within the range of investigated material states.

The pendular state to which our model applies is only relevant in some limited saturation range. On the one hand, a minimum liquid volume is necessary for menisci to form at contacts, as the liquid will first cover the grain surface asperities. This minimum saturation S_{\min} for bridges to form might be roughly estimated upon introducing a roughness scale δ , assuming a layer of thickness δ covers the surface of the grains, as

$$S_{\min} = \frac{6\Phi\delta}{(1 - \Phi)a}. \quad (6)$$

For $\Phi = 0.5$ and $\delta \sim 10^{-4}a$ the minimum value for saturation is of the order of 10^{-3} , as observed in experiments [17]. Using (5), and typical values of z (5 or 6) and Φ (say, 0.6), this sets a lower bound to meniscus volume, of order $10^{-4}a^3$. On the other hand, the upper saturation limit for the pendular state corresponds to

the merging of the menisci pertaining to the same grain, which, considering a triangle of spherical grains in mutual contact, happens as soon as the filling angle (see Fig. 1) reaches $\pi/6$. The analytical formula for V [34], within the toroidal approximation, as a function of φ (setting $h = 0$, and $\theta = 0$), then yields $\frac{V}{a^3} \simeq 8.10^{-3}$, corresponding, using (5), to a maximum saturation between 0.05 and 0.1, similar to experimental observations [17, 18].

C. Choice of parameters

Tab. I gives the values of parameters employed in our simulations. While stiffness number κ and friction coefficient μ are kept fixed, reduced pressure varies from the dry case $P^* = \infty$ down to the lowest value 0.1, for which cohesive effects are strong, while the investigated range of I values allows us to approach the quasistatic limit with some accuracy, as well as assess the effects of inertia in faster flows (although rapid, strongly agitated flow are not studied here). The meniscus volume is chosen as $V = 10^{-3}a^3$ in most simulations. A few tests are carried out with different values (as given within brackets) of the number of particles, the stiffness number and the preset meniscus volume. Taking $\Gamma = 7.3 \times 10^{-2} \text{ J.m}^{-2}$

κ	8400 (occasionally 39000)
μ	0.3
N	4000 (8000)
I	from 10^{-4} to 0.562 by factors of $\sqrt{10}$
P^*	0.1 ; 0.436 ; 1 ; 2 ; 5 ; 10 ; ∞
V/a^3	10^{-3} (10^{-2} ; 5×10^{-3} ; 2×10^{-4} ; 10^{-6})

TABLE I: List of parameter values: N particles of diameter a , interacting with friction coefficient μ , forming menisci of volume V at contacts, are subjected to normal stress-controlled shear flow for which inertial number I , reduced pressure P^* (evaluated with normal stress σ_{22}) and stiffness parameter κ take values as prescribed. Attractive forces fall to zero at distance $D_0 = V^{1/3}$.

for water, and $a = 0.1 \text{ mm}$, $P^* = 1$ corresponds to $\sigma_{22} = (\pi\Gamma/a) \simeq 2.3 \text{ kPa}$ – the pressure, under gravity, below a granular layer with a thickness of a few tens of centimeters.

III. HOMOGENEITY AND STATIONARITY

A. Steady states, macroscopic measurements

Starting from a dense initial configuration, with solid fraction close to the random close packing value ($\Phi_{\text{RCP}} \simeq 0.64$), we impose a constant shear rate $\dot{\gamma}$ and wait until a steady state is reached before measuring constitutive relations for stresses and solid fraction, which are identified

as averages over time series. Stresses are measured using the standard formula, for all coordinate index pairs α, β

$$\sigma_{\alpha\beta} = \frac{1}{\Omega} \left[\sum_i m v_i^\alpha v_i^\beta + \sum_{i<j} F_{ij}^\alpha r_{ij}^\beta \right], \quad (7)$$

involving a kinetic contribution with a sum over grains i , of velocities v_i and a sum over all pairs with center-to-center vector r_{ij} , interacting with force F_{ij} , Ω denoting the sample volume.

The evolution of solid fraction Φ with strain γ is shown in Fig. 3: Φ decreases until it approaches its steady state value for $\gamma \geq 5$ in this case. Fig. 4 shows the evolutions of σ_{22} and σ_{12} with γ . (Note that σ_{12} is negative with our sign convention). We thus check that normal stress σ_{22} is well controlled since its fluctuations about its prescribed value P are very small. Shear stress σ_{12} exhibits a fast increase and overshoot at small strain and then decreases, approaching its steady state value, after a few strain units, over a strain interval (a few units) similar to the one corresponding to the transient evolution of Φ . Stresses and solid fraction fluctuate in the steady state, and a careful evaluation of measurement errors on their time averages is required (especially for shear stress, for which fluctuations levels reaching about 20% of the mean value are apparent in the example of Fig. 4). We use the blocking technique of Ref. [42] to estimate error bars on averages over finite time series.

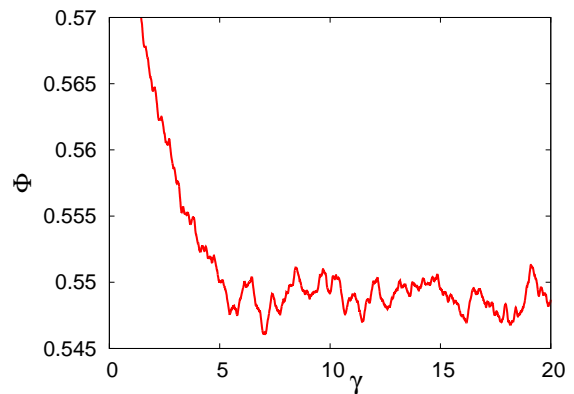


FIG. 3: (Color online) Solid fraction Φ versus shear strain γ . Time series is obtained with $P^* = 1$, $I = 0.1$ and $N = 4000$ when the rupture distance is $D_0 = 0.1$.

B. Shear localization

1. Velocity profile

Instantaneous velocity profiles $v^s (\equiv \langle v_1(x_2) \rangle_s)$ are computed on averaging particle velocities along the mean flow direction within slices of thickness $0.01L_2$. We observe a strong localization of the flow for the smallest

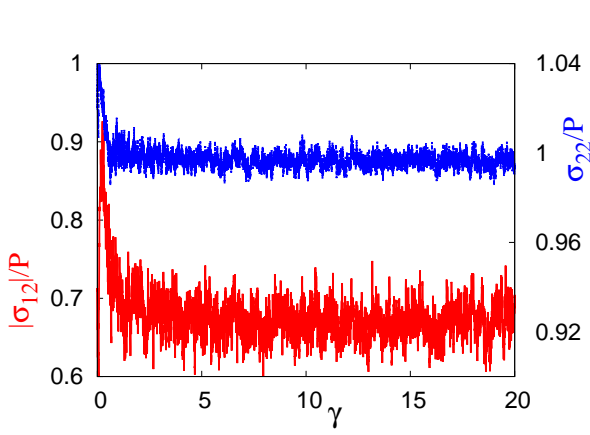


FIG. 4: (Color online) Shear stress $|\sigma_{12}|$ (lower curve, red, left axis) and normal stress σ_{22} (upper curve, blue, right axis) versus shear strain γ . Note the different scales on left and right axes. Time series obtained with $P^* = 1$, $I = 0.1$ and $N = 4000$ when the rupture distance is $D_0 = 0.1$.

value of the reduced pressure, $P^* = 0.1$, for both slow and flows. As represented in Fig. 5a, the velocity gradient, initially uniform, gradually concentrates within a shear band of thickness $H \lesssim 3a$ which may move vertically but persists for all values of strain $\gamma > 250$. Localization tendencies in slow flow of dry granular materials are sometimes reported [1, 5, 43], although, for uniform strain rates, usually not observed as an enduring, systematic phenomenon. In the present study persistent localization profiles are also detected in nearly quasistatic flows, with a shear band thickness between $5a$ to $10a$. However, for the intermediate values of the inertial number ($10^{-2} \lesssim I \lesssim 10^{-1}$) this effect diminishes and the strongly localized profiles are less frequent.

For all $P^* \geq 0.436$, localization is not frequently observed and temporary, even in the quasistatic limit. The velocity profiles for $P^* = 0.436$ and $I = 10^{-3}$, as represented in Fig. 5b, are nearly linear and on average the flow is homogeneous.

2. Local solid fraction

Similarly we can calculate a solid fraction profile, on averaging the solid contents of slices orthogonal to the velocity gradient (splitting the volume of one grain between different slices if necessary). Fig. 6 shows the velocity (v^s) and solid fraction (Φ^s) profiles for two different values of shear strain, $\gamma = 1$ and $\gamma = 352$, which belong to the simulation of Fig. 5a. We see that in the homogeneous flow the distribution of mass in the system is almost uniform, but when the localization occurs Φ^s strongly decreases within the shear bands, to a value below 0.2. It slightly increases outside the shear band, especially in its vicinity. Slighter decrease of density within thicker shear bands in quasistatic flow is observed. For

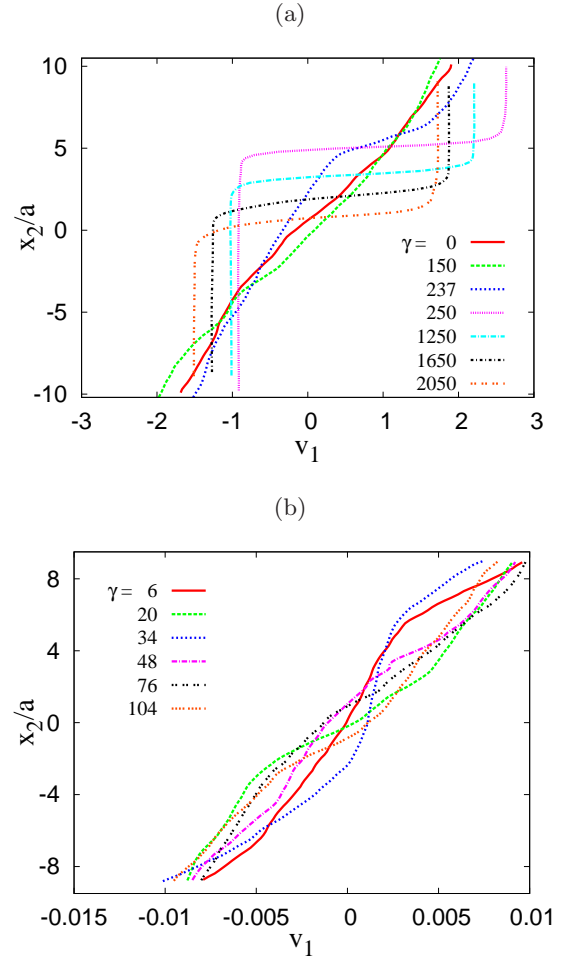


FIG. 5: (Color online) Velocity profile for $P^* = 0.1$, $I = 0.178$ (a) and for $P^* = 0.436$ and $I = 10^{-3}$ (b) at different shear strain values.

instance, when $I = 10^{-3}$, Φ^s decreases from 0.47 to about 0.4 inside the sheared zone of thickness $H \approx 7a$.

3. Deviation from linear profile

The deviation from the linear profile is characterized by parameter Δ :

$$\Delta(t) = \frac{12}{L_2^3 \dot{\gamma}^2} \int_{-L_2/2}^{L_2/2} (v_1(x_2) - \dot{\gamma} x_2)^2 dx_2. \quad (8)$$

The normalization by $\frac{L_2^2}{12}$ ensures a maximum value $\Delta = 1$ in the case of a perfect localization within a plane, as if two solid blocks were sliding on each other. As defined in Eq. (8), parameter Δ is not affected by a vertical shift in the velocity profile, due to the periodic boundary condition in direction x_2 . If the strain rate is homogeneous within a shear band of thickness H and vanishes

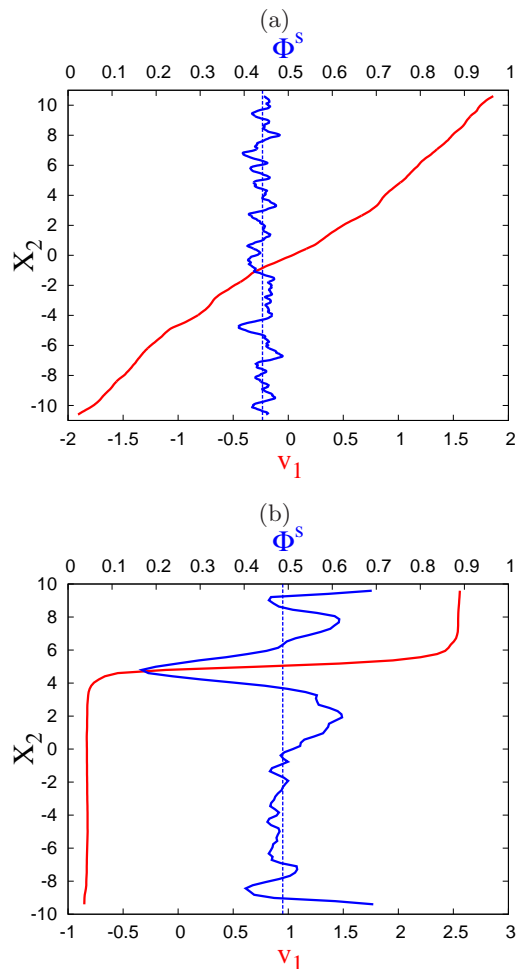


FIG. 6: (Color online) Velocity profiles (lower axis in red) and local densities of grains (upper axis in blue) for two configuration samples with shear strain $\gamma = 20$ (a) and $\gamma = 352$ (b) when $P^* = 0.1$ and $I = 0.178$. Average solid fraction $\langle \Phi^s \rangle$, is shown with a vertical dashed line in blue, with a value of 0.44 in (a) and 0.49 in (b).

outside, one observes

$$\Delta = \left(1 - \frac{H}{L_2}\right)^2 \quad (9)$$

Fig. 7 is the plot of Δ as a function of strain γ corresponding to the same simulation as in Fig. 5a. It initially shows small fluctuations near zero, and suddenly increases near $\gamma = 250$ when the velocity gradient localizes in a shear band.

4. Occurrence of shear banding

Large values $\Delta > 0.8$ for $P^* = 0.1$ in the faster flows ($I \geq 0.17$) indicate strong localization in this range. At $I = 0.1$ Δ drops down to small values, typically below 0.1, but in the quasistatic limit it increases again: at

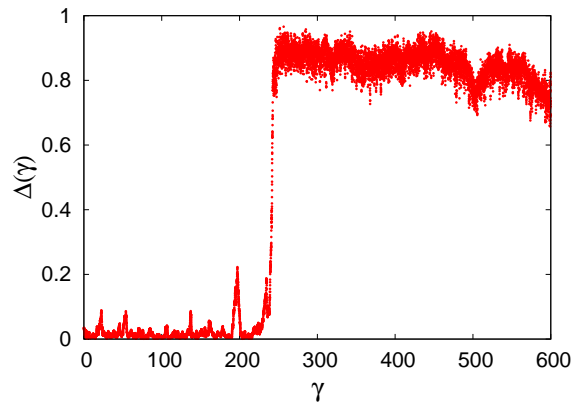


FIG. 7: (Color online) Deviation from linear profile, $\Delta(\gamma)$, versus strain γ for $P^* = 0.1$, $I = 0.178$.

$I = 10^{-3}$ it mainly fluctuates between 0.4 and 0.8. For $P^* \geq 0.436$ the shear rate is much more homogeneous. Δ almost vanishes in faster flows, increases somewhat in the quasistatic limit, but rarely exceeds 0.2, even for the smallest inertial numbers.

Simulations carried out with a larger stiffness number ($\kappa = 39000$), for the two smallest values of P^* and for all values of I in Tab. I, do not record any significant influence of κ on the homogeneity of the flow. The influence of the sample size is studied by simulating some samples with height L_2 twice as large as in the standard sample, containing 8000 grains, with $P^* = 0.436$ and different values of I . The size dependence in formula 9 implies then larger values of Δ , should the shear strain tend to localize, temporarily or permanently, within a region of fixed thickness. In our tall, 8000 grain systems, as the quasistatic limit is approached, Δ reaches peak values above 0.4 but continuously evolves and no persistent localization pattern is detected.

Consequently, our results reveal a strong localization tendency at $P^* = 0.1$ for both small (below 0.03) and large (above 0.3) values of the inertial number. We performed some measurements at $P^* = 0.1$ for intermediate values of I , over strain intervals for which values of inhomogeneity parameter Δ averaged below 0.1, as in the first part of the graph of Fig.7. We limited our systematic studies to homogeneous flows, for larger values of P^* , for which no evidence of enduring localization effects is observed.

A systematic fluid depletion in shear bands was reported in [44] – this requires a model for liquid migration between menisci, which we did not introduce in the present study. We limit our results to the issue of whether shear banding occurs, and focus on homogeneous flows. Shear banding and inhomogeneous flows certainly deserve more detailed studies, which we plan to pursue in another publication.

IV. MACROSCOPIC BEHAVIOUR AND CONSTITUTIVE RELATIONS

Focussing on homogeneous steady states (excluding too small P^* values, from Sec. IIIB) we now deduce macroscopic constitutive relations from the simulations.

A. Shear stress and solid fraction

Friction coefficient μ^* and solid fraction Φ , depending on I for various P^* values, are shown in Fig. 8 for the parameter choice adopted in most simulations. We fit the

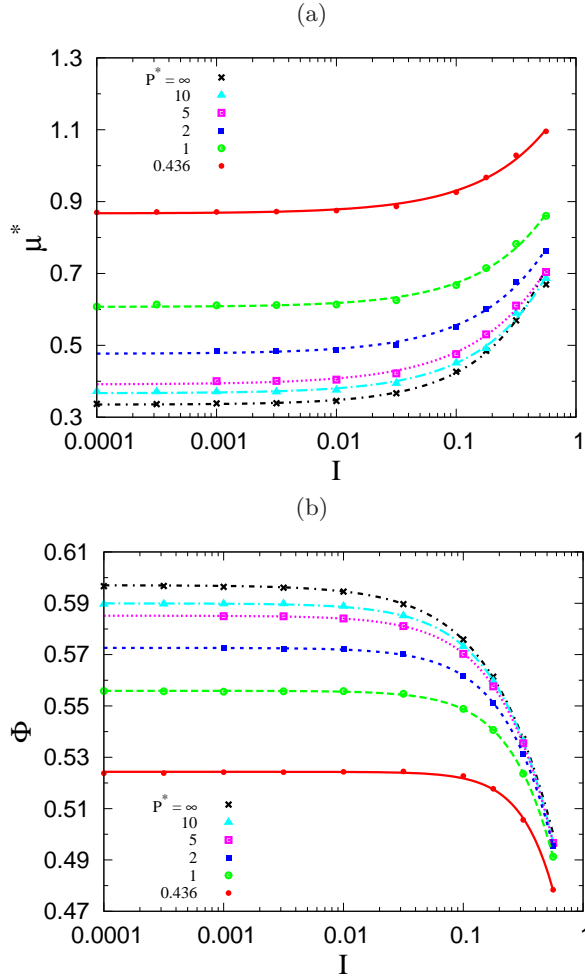


FIG. 8: (Color online) Macroscopic friction coefficient μ^* (a) and solid fraction Φ (b) versus inertial number I for different values of reduced pressure P^* ($D_0 = 0.1a = V^{1/3}$).

following power law functions to those data, denoting as μ_0^* and Φ_0 the quasistatic limits (critical state) values of the macroscopic friction coefficient and of the solid fraction:

$$\begin{cases} \mu^* = \mu_0^* + cI^\alpha \\ \Phi^{-1} = \Phi_0^{-1} + eI^\nu \end{cases} \quad (10)$$

Tabs. II and III give the values of the fitting parameters introduced in Eqs. 10. While the increase of μ^* and the decrease of Φ as functions of I are familiar trends, similar to observations made with dry grains [1, 4–6], some other features are remarkable. The quasistatic limit is quite nearly approached for $I \leq 0.01$, and is strongly influenced by capillary forces. Internal friction coefficient μ^* , compared to the dry, cohesionless value (0.335 ± 0.001), already shows a notable increase at $P^* = 10$, reaching values as high as 0.6 for $P^* = 1$ (*i.e.*, as cohesive and confining forces are of the same order), and nearly 0.9 for $P^* = 0.436$, about 2.3 times the cohesionless value. Our partial results for $P^* = 0.1$, measured in reasonably homogeneous flows, indicate $\mu^* \simeq 1.6$ for $I = 10^{-2}$. Meanwhile, the material becomes looser, with Φ reaching values that cannot be observed without cohesion in quasistatic conditions.

P^*	μ_0^*	α	c
0.436	0.867 ± 0.003	0.76 ± 0.04	0.36 ± 0.01
1	0.607 ± 0.003	0.79 ± 0.05	0.41 ± 0.02
2	0.477 ± 0.005	0.76 ± 0.05	0.45 ± 0.02
5	0.391 ± 0.006	0.73 ± 0.05	0.49 ± 0.02
10	0.367 ± 0.004	0.79 ± 0.04	0.51 ± 0.02
∞	0.335 ± 0.001	0.84 ± 0.02	0.62 ± 0.02

TABLE II: Parameters of the fit of function $\mu^*(I)$ by Eq. 10, for different values of P^* .

P^*	Φ_0	ν	e
0.4360	0.5243 ± 2.10^{-4}	1.73 ± 0.05	0.497 ± 0.017
1	0.5559 ± 10^{-4}	1.34 ± 0.012	0.512 ± 0.005
2	0.5726 ± 10^{-4}	1.21 ± 0.01	0.547 ± 0.003
5	0.5851 ± 10^{-4}	1.12 ± 0.01	0.580 ± 0.003
10	0.5900 ± 10^{-4}	1.09 ± 0.01	0.594 ± 0.004
∞	0.5970 ± 10^{-4}	0.96 ± 0.015	0.562 ± 0.008

TABLE III: Parameters of the fit of function $\Phi(I)$ by Eq. 10, for different values of P^* .

Such a strong influence of cohesive (capillary) forces contrasts with the results of Refs. [11, 16], in which similar deviations between cohesionless and cohesive systems are not observed until P^* decreases to much lower values, of order 0.01. Such 2D results were however obtained with a different attractive force law, of vanishing range beyond contact. The origins of the strong rheological effects of capillary forces are investigated in the following.

B. Normal stress differences

The first and the second normal stress differences are defined as

$$\begin{cases} N_1 &= \sigma_{11} - \sigma_{22} \\ N_2 &= \sigma_{22} - \sigma_{33} \end{cases} \quad (11)$$

Note that those definitions coincide with the one used in complex fluid or suspension rheology [45], but that we use the opposite sign convention for normal stresses. Signs of N_1 and N_2 should thus be reversed for comparisons to this literature.

Most often, considering dense flows of dry granular materials, those differences, deemed small, are ignored or neglected [29]. We find it worthwhile to record their values nevertheless, since, as shown in Fig. 9, where N_1 and N_2 are plotted versus I for different values of P^* , they are strongly influenced by capillary forces. The first normal stress difference is very small in the quasistatic limit and for large values of the reduced pressure. It increases with I and for decreasing values of P^* , going through a transition from small negative values to positive values near $I = 0.03$, for $P^* \geq 2$. N_1 variations with I are nearly parallel for different P^* values. The second normal stress difference N_2 also increases for faster flows and for decreasing reduced pressure P^* . In the quasistatic limit, it is considerably larger than N_1 .

C. Sensitivity to capillary force model and saturation

1. Capillary force model

We tested the effect of the capillary force model by replacing the Maugis approximation, Eq. 4, with the more accurate parametrized capillary force law proposed by Soulié *et al.* [20, 46], for $V = 10^{-3}a^3$. Although the difference in the force models is appreciable on a plot of F^{Cap} versus h (with the Soulié force about 10% smaller at contact, see Fig. 2), the macroscopic results are very close: the difference in stress ratio μ^* and solid fraction Φ increases with I but does not exceed 2%.

2. Meniscus volume and force range

Changing the meniscus volume amounts to changing the distance at which the attractive force vanishes, rupture distance $D_0 = V^{1/3}$, as well as the gap dependence of the capillary force $F^{\text{Cap}}(h)$ (Fig. 2). Fig. 10 shows internal friction coefficient $\mu^*(I)$ to be strongly sensitive to meniscus volume for the lowest P^* values. For a meniscus volume of $10^{-6}a^3$, as compared to the standard value $10^{-3}a^3$, μ^* decreases by about 20%. Actually, for such a small meniscus volume, the decay of the attractive force (Fig. 2) is so fast that, as we checked, results are hardly

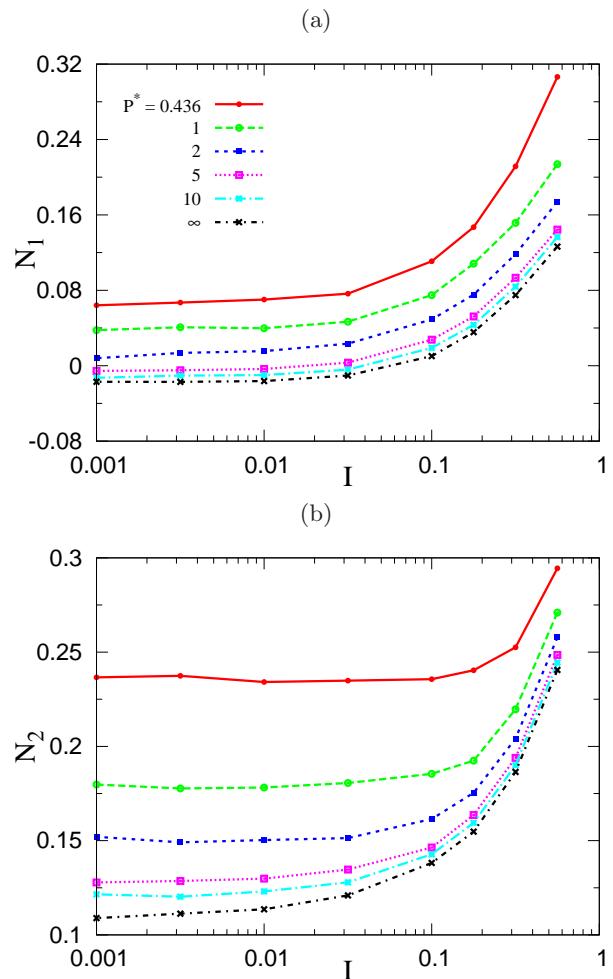


FIG. 9: (Color online) First (a) and second (b) normal stress differences as functions of I for different P^* values. The same symbols and colour codes apply to both figures.

changed on setting the meniscus rupture distance to zero. The effect on the solid fraction Φ remains small. To explore the rheological properties throughout the pendular regime, we varied the meniscus volume, and recorded the solid fraction and the friction coefficient in the quasistatic limit for the smallest studied P^* value, as indicated in Tab. IV, thus fully covering the corresponding saturation range (see Sec. II B). Saturation S , by relation (5), is related to the wet coordination number, z , whose values are also provided in the table. While the change in solid fraction does not exceed 0.01, the variation of the macroscopic friction coefficient is about 20% in the pendular regime (up to 50% upon extending the numerical study to unrealistically small menisci, $V = 10^{-6}a^3$). We therefore predict a moderate variation of rheological properties within the simulated pendular regime of the partially saturated granular assembly. Returning to the basic assumptions of our model, one of its drawbacks is that it ignores liquid volume conservation. Within the granular sample, the total liquid volume is proportional

3. Hydraulic hysteresis

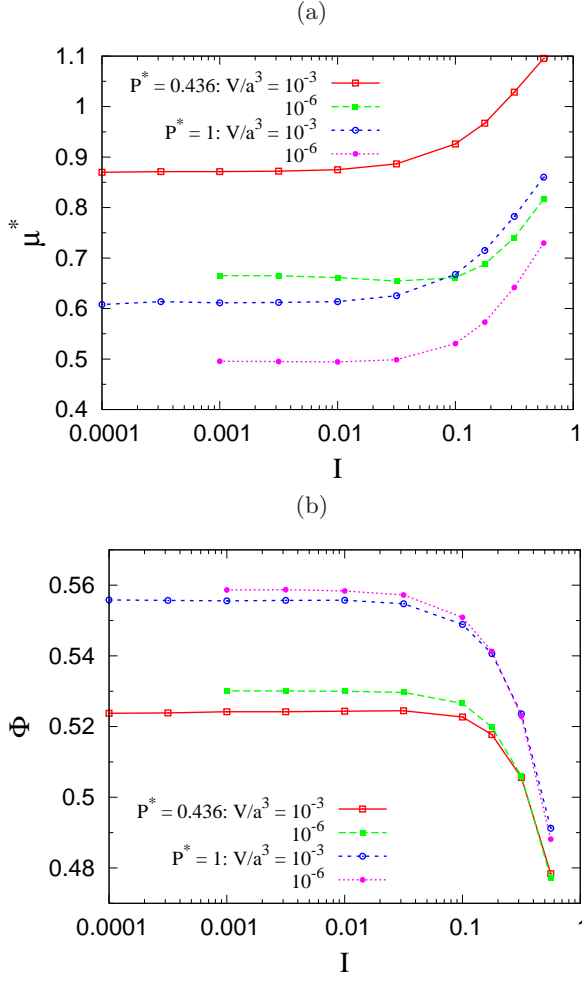


FIG. 10: (Color online) Macroscopic friction coefficient μ^* (a) and solid fraction Φ (b) versus inertial number I for different values of P^* and meniscus volume V (with $D_0 = V^{1/3}$).

V/a^3	S_w	z	Φ	μ^*
10^{-2}	7.137×10^{-2}	6.863	0.520	1.071
5×10^{-3}	3.418×10^{-2}	6.556	0.522	1.003
10^{-3}	6.305×10^{-3}	5.970	0.524	0.875
2×10^{-4}	1.075×10^{-3}	5.534	0.525	0.787
10^{-6}	5.539×10^{-6}	4.836	0.530	0.661

TABLE IV: (Color online) Effect of meniscus volume or saturation level on different parameters for $I = 10^{-2}$ and $P^* = 0.436$.

to coordination number z . As z varies with I , we should in principle correct the meniscus volume to maintain a constant product zV for different shear rates. However, the V dependence of macroscopic properties is so slow (μ^* varies by 20% as V is multiplied by 20) that the resulting correction on V (as z changes, typically, from 6 to 4 at most) should not exceed 2%.

Another feature of meniscus model the role of which should be explored is the hysteresis of the attractive force, which appears at contact, and vanishes at distance D_0 . Given the results of Tab. V, one may expect a strongly enhanced influence of distant interactions (as reported, e.g., in Ref. [47]). Fig. 11 compares internal friction and solid fraction, for different values of P^* and I , in the standard, hysteretic model, and without the capillary force hysteresis, assuming menisci to appear as soon as non-contacting grains approach below distance D_0 . With the

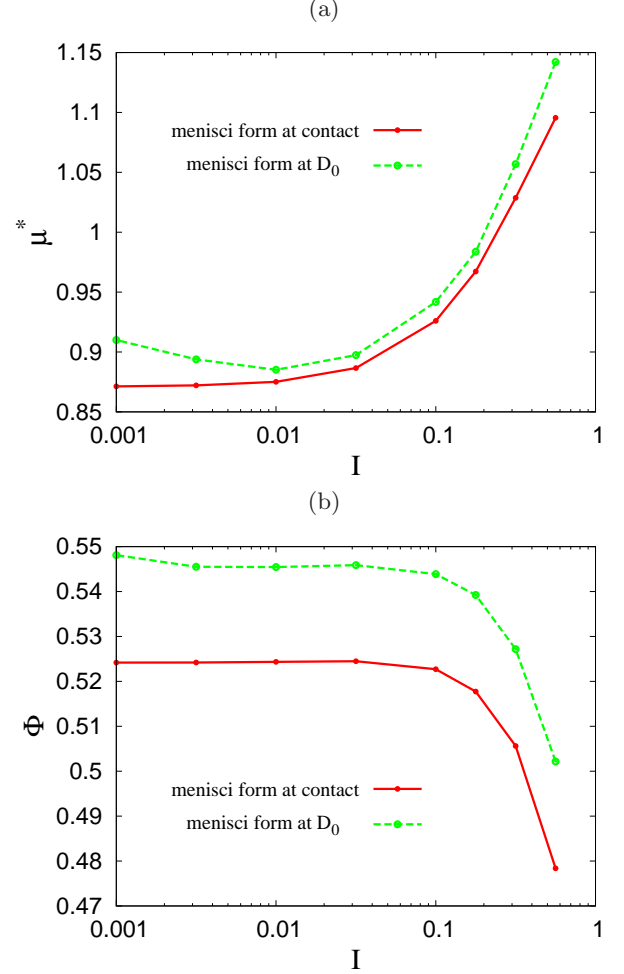


FIG. 11: (Color online) Macroscopic friction coefficient μ^* (a) and solid fraction Φ (b) versus inertial number I for $P^* = 0.436$ and $D_0 = 0.1$, with and without capillary hysteresis.

new rule of meniscus formation, Φ strongly increases, especially for small values of I . The internal friction μ^* , for $I \simeq 0.1$, is close to the standard case, but larger values are obtained as I decreases. Even for the smallest values of I investigated ($I = 0.001$), the material properties still depend on shear rate and no proper critical state appears to be approached in our simulations. The decrease of μ^* as a function of I in interval $0.001 \leq I \leq 0.01$ should trig-

ger shear-banding instabilities, as discussed in [48, 49]. A slightly decreasing trend of μ^* versus I was also apparent in Fig.10, for very small D_0 . For the standard value $V = 10^{-3}a^3$ adopted in this study (as one for which laboratory observations should be possible), the friction coefficient does increase with I , albeit slower and slower as P^* decreases (see coefficient c in Tab.II). The stabilizing effect of this growing variation is weaker as cohesion gets stronger, which is consistent with the systematic shear banding behaviour at $P^* = 0.1$, and might be jeopardized on tampering with the capillary force model.

V. RHEOLOGICAL EFFECT OF CAPILLARY FORCES

We now seek to explain the strong influence of capillary forces on the macroscopic material rheology. The roles of different interactions, in the force network and in the stresses are investigated. We first collect information on coordination numbers and neighbor distances (Sec.V A). Simple relations to average forces are recalled in Sec. V B. We split the stresses into several contributions, in order to appreciate the importance of different types of forces. This decomposition (Sec. V C) suggests an attempt to relate the rheology of wet grains to that of dry ones, in terms of some "effective pressure" approach, in the quasistatic limit, which we present in Sec. V D.

A. Coordination numbers and near neighbor distances

Fig. 12 shows the variations, with inertial number I , for different P^* values, of coordination numbers z_c , for pairs of grains in contact, and z_d , for pairs of grains attracting each other without contact at a distance lower than D_0 . The average number of contacts per grain, z_c , decreases for larger inertial numbers, as previously observed in cohesionless systems [1, 5] and in cohesive ones [16], slower for smaller P^* , as in [16] too. z_c also increases as P^* decreases at constant I , as previously observed as well [16]. Note that this latter trend is opposite to that of the solid fraction (Fig. 8). As the importance of adhesion, relative to confinement stresses, increases, looser systems are obtained, yet better coordinated. Grains tend to stick to one another, and may form loose aggregates, as in static or quasistatically compressed assemblies, for which little correlation is also observed [13, 15] between density and coordination number. On the other hand, the variations of the coordination number of distant interactions, z_d , with both parameters I and P^* , are in the opposite direction to those of z_c . As I increases, so does z_d : contacting pairs tend to separate, but some remain bonded by liquid bridges. And for stronger cohesion (smaller P^*), z_d is correlated with the system density. The faster approach to quasistatic limit at smaller values of P^* is apparent in both figures. The fraction of *rattlers* (beads carrying no

force [5]) in non-cohesive systems is about 5%. In the cohesive case, due to the attractive forces, nearly all of the particles are bonded to others and the number of rattlers tends to zero, as observed in 2D simulation of cohesive powders [13, 15]. z_d tends to compensate the changes of

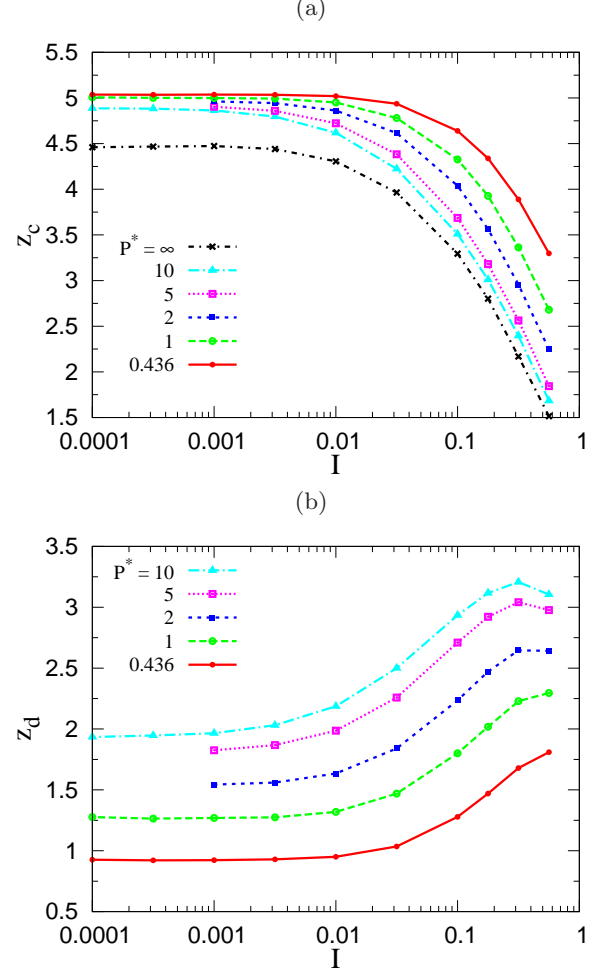


FIG. 12: (Color online) Coordination numbers: (a) of contacts, z_c ; (b) of distant interactions, z_d .

z_c , so that the total coordination number $z = z_c + z_d$, throughout the investigated range of I and P^* values, exhibits rather small variations (see Fig. 13). Within the investigated parameter range, the maximum change in z , between 6.8 and 4.8, corresponds to a correction of internal friction μ^* , should we change the meniscus volume to maintain the total liquid volume constant, of about 5% (see Tab. IV). The contact coordination number does not change much with the force range or the meniscus volume. Setting $D_0 = 0$ (instead of the standard value 0.1 used in the present study) or decreasing the volume of the meniscus from its standard value $V = 10^{-3}a^3$ down to $10^{-6}a^3$, merely leads to a small decrease of z_c , from 5 to 4.7 in the quasistatic limit, when $P^* = 0.436$. However, it has a strong influence on z_d . Compared to the standard case, for $P^* = 0.436$ and small values of I , it

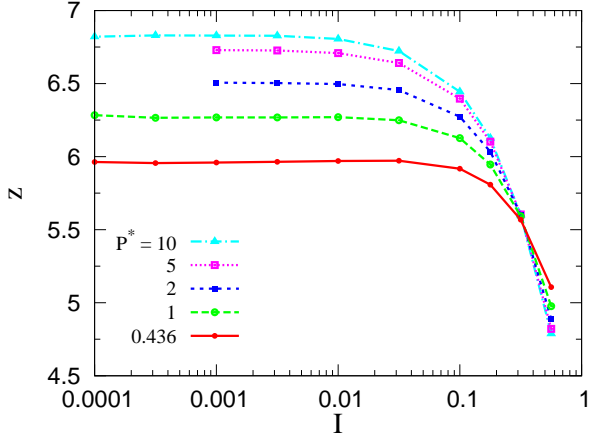


FIG. 13: (Color online) Total coordination number $z = z_c + z_d$.

decreases from 0.9 down to a value below 0.3 when we set $D_0 = 0.01$, or down to about zero when we set $V = 10^{-6}$.

It is interesting to compare the number of distant, interacting pairs to the total number of neighbor pairs at distance below D_0 . The coordination number, $z(h)$, of neighbor grains at distance below h (such that $z(0) = z_c$) grows with h as depicted in Fig. 14, corresponding to $I = 10^{-3}$ (quasistatic limit). $z(h)$, like the contact coordination number, is a decreasing function of P^* for small h/a (below about 2.5×10^{-3} , see the insert in Fig. 14). It increases with P^* , like the density, beyond that distance. In denser systems grains have more neighbors on average, but this is only true if neighbors at some distance are included in the count, and does not apply to contacts (a situation reminiscent of some observations in static packings of cohesionless grains [31]). Up to meniscus rupture distance D_0 , equal to $0.1a$ in the present case, each grain has on average $z(D_0) - z_c$ non-contacting neighbors, among which z_d are joined by a liquid bridge. Values of ratio $z_d/(z(D_0) - z_c)$ for different P^* and I are given in Tab. V. The proportion of the neighbors within range D_0 that are bonded by a liquid bridge varies from 0.61 to 0.71 for $I = 10^{-3}$ and between 0.68 and 0.79 for $I = 10^{-1}$ – slightly larger than the proportion $\sim 50\%$ reported by Kohonen *et al.* [33] in static grain packs.

	$I = 10^{-3}$			$I = 10^{-1}$	
P^*	z_d	$z_d/(z(D_0) - z_c)$		z_d	$z_d/(z(D_0) - z_c)$
0.436	0.923	0.609		1.28	0.681
1	1.27	0.650		1.80	0.723
2	1.54	0.675		2.24	0.751
5	1.83	0.701		2.71	0.776
10	1.97	0.712		2.93	0.787

TABLE V: Distant coordination number z_d and proportion of pairs within distance D_0 joined by a meniscus, $z_d/(z(D_0) - z_c)$, versus P^* for two different values of I .

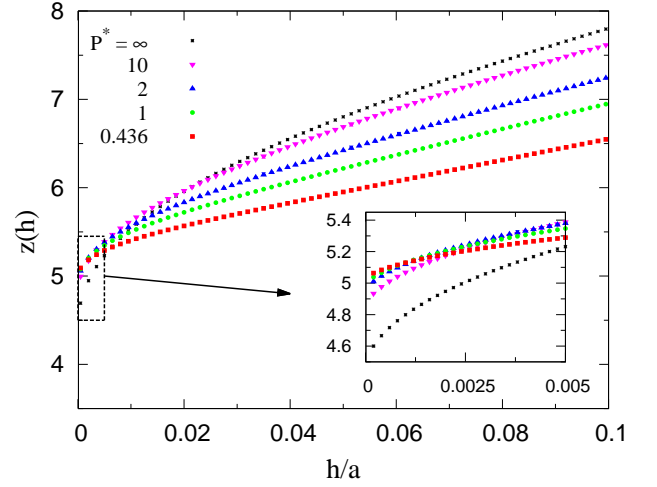


FIG. 14: (Color online) Coordination number of neighbor grains versus interparticle distance h , for different P^* values and for $I = 10^{-3}$. Inset shows detail at small h .

If the meniscus forms as soon as grains approach at distance D_0 , rather than at contact, the number of contacts hardly changes (z_c increases by about 5% for $P^* = 0.436$ in the quasistatic limit), but the increase in the number of menisci is larger than expected from the data of Tab. V, from a simple count of pairs within range D_0 : z_d is multiplied by 1.7 at small P^* and I .

B. Pressure and average normal forces

From Eq. 7, neglecting the deflection of contacts in comparison to grain diameter a , and ignoring the kinetic term, one may relate [31] the average pressure, $\mathcal{P} = \text{tr} \underline{\underline{\sigma}}/3$, to the average normal force $\langle F^N \rangle$ for all interactions, and to the average, $\langle F^N h \rangle_d$, over pairs in distant interactions, of the product of force by distance $h \leq D_0$:

$$\mathcal{P} = \frac{\Phi z}{\pi a^2} \langle F^N \rangle + \frac{\Phi z_d}{\pi a^3} \langle F^N h \rangle_d \quad (12)$$

Due to normal stress differences, ratio $\frac{\mathcal{P}}{\sigma_{22}}$ is only slightly different from 1 (about 0.95) at small I . We checked that formula (12) is very accurate for all P^* values, and found its second term to be negligible, contributing less than 2% of the pressure.

C. Contributions to stresses

The contribution of the kinetic term in Eq. 7 to stresses is quite small. Even for the fastest flow in our simulation ($I = 0.562$), this contribution does not exceed 2% of the shear stress or 5% of the normal stress components, and for $I = 0.178$ it is nearly zero for all stress components.

Therefore, in this section we only discuss the contributions of forces to the stress components, for the different values of the control parameters, P^* and I . These contributions may be split in different ways, on distinguishing different forces.

1. Contact forces and distant capillary attraction

First one may consider the total stress as a sum of the contributions of the contacts and of the distant interacting pairs, as

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^c + \sigma_{\alpha\beta}^d. \quad (13)$$

Our results show that the contribution of contact forces dominate in the shear stress. It is larger than 90%, regardless of the values of P^* and I . The contribution of distant interactions to σ_{12} , as represented in Fig. 15 a, is not negligible and increases for smaller values of P^* . However, it hardly exceeds 10% of the total shear stress.

The contribution of distant interactions to σ_{22} is displayed in Fig. 15b. Capillary forces being attractive, σ_{22}^d is a tensile stress. For $P^* = 0.436$, in the quasistatic limit, this contribution increases up to 20% in magnitude. Consequently, the positive contribution of contact forces to σ_{22} reaches about $1.2\sigma_{22}$ for $P^* = 0.436$.

The relative importance of the contributions of contacts and distant capillary forces to σ_{12}^d and σ_{33}^d is similar: in the quasistatic limit and for $P^* = 0.436$, one has $\sigma_{11}^d/\sigma_{11} \simeq -0.16$ and $\sigma_{33}^d/\sigma_{33} \simeq -0.25$.

Accordingly, the contribution of normal contact forces is the dominant one in normal stress differences N_1, N_2 (with a notable contribution of tangential forces to N_2 , typically 20% at low P^*).

2. Elastic-frictional forces and capillary forces

An alternative decomposition of the stress tensor is:

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^{\text{cap}} + \sigma_{\alpha\beta}^{\text{Ne}} + \sigma_{\alpha\beta}^{\text{T}}, \quad (14)$$

in which $\sigma_{\alpha\beta}^{\text{cap}}$ is the contribution of capillary forces (either in the contacts or for distant interacting pairs), $\sigma_{\alpha\beta}^{\text{Ne}}$ is the contribution of normal elastic forces and $\sigma_{\alpha\beta}^{\text{T}}$ is the contribution of tangential forces.

The normal elastic forces contribute more than 90% of the shear stress, whatever P^* and I .

The contribution of tangential forces to the normal (diagonal) elements of the stress tensor is negligible, but that of capillary forces is very important: for $P^* = 0.436$, negative terms $\sigma_{\alpha\alpha}^{\text{cap}}$ ($1 \leq \alpha \leq 3$) are very large in magnitude: one observes $\sigma_{\alpha\alpha}^{\text{cap}} < -2\sigma_{\alpha\alpha}$ for small P^* , as shown in Fig. 16. This large negative contribution is compensated by that of the repulsive normal elastic forces, $\sigma_{\alpha\alpha}^{\text{Ne}} > 3\sigma_{\alpha\alpha}$. Such a large negative contribution of capillary forces to pressure implies that the particles are

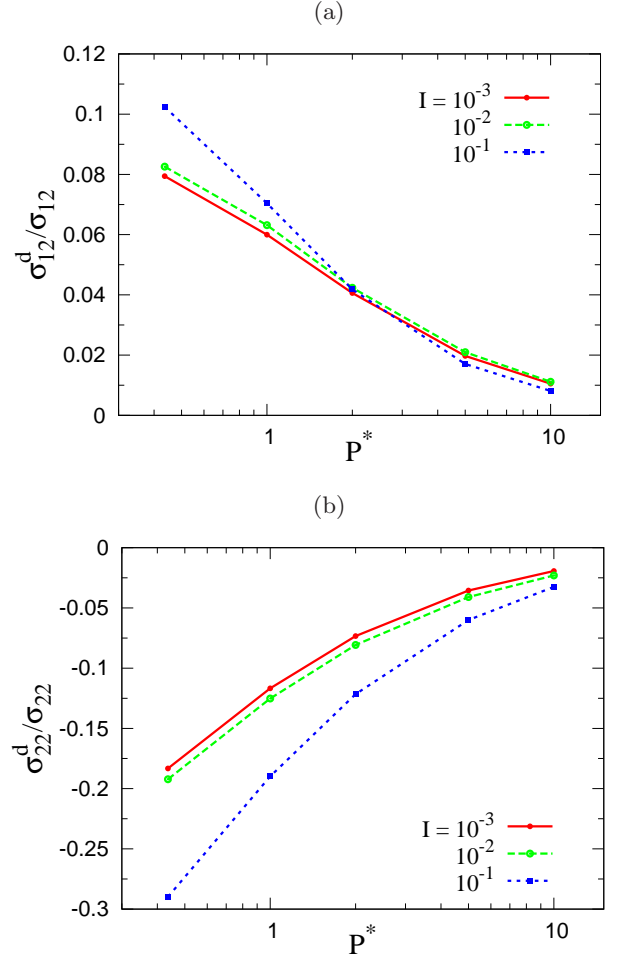


FIG. 15: (Color online) Contribution of distant interactions to shear stress σ_{12} (a) and to normal stress σ_{22} (b).

strongly pushed against one another, which increases the sliding threshold for tangential contact forces.

Fig. 17 shows the contribution of tangential forces to the total shear stress. As P^* is decreased to $P^* = 0.436$, the ratio $\sigma_{12}^{\text{T}}/\sigma_{12}$ increases to 0.18. Capillary forces contribute to the shear stress with the opposite sign (σ_{12}^{cap} is positive while σ_{12} is negative). Fig. 17 shows that ratio $\sigma_{12}^{\text{cap}}/\sigma_{12}$ is always negative and decreases down to -0.12 for $P^* = 0.436$. Similarly to the case of normal stresses, the largest contribution is that of elastic normal forces, the (negative) contribution of which to σ_{12} compensates the (positive) term σ_{12}^{cap} .

D. Discussion

One important clue to understand the enhanced shear strength of the cohesive material, as compared to the cohesionless, dry granular assembly, is the large tensile contribution of capillary force to normal stress:

$$\sigma_{22}^{\text{cap}} = -\beta\sigma_{22}, \quad (15)$$

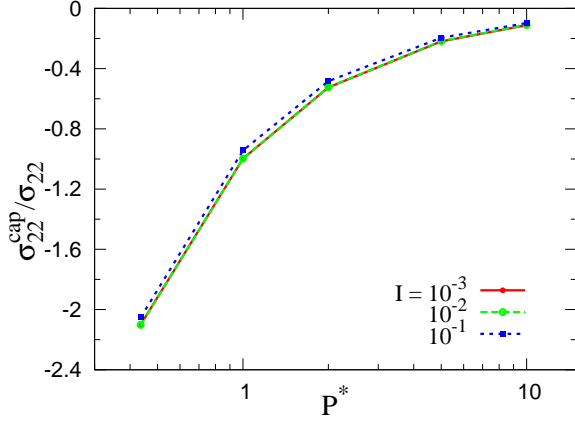


FIG. 16: (Color online) Contribution of capillary forces to stress σ_{22} .

with a coefficient β ranging, in the quasistatic limit, from about 0.15 ($P^* = 10$) to 2.1 ($P^* = 0.436$). Upon including the result for $P^* = 0.1$ and $I = 0.01$, β reaches about 7.2. This coefficient, and its variations with P^* , can be approximately predicted from the values of solid fraction and coordination numbers. Contacts (z_c , on average, per grain) carry capillary force $-F_0$, and distant forces (z_d per grain) average to a fraction of $-F_0$. Relation (12) can be used to evaluate the capillary contribution to pressure \mathcal{P} , as $-\frac{\Phi z F_0}{\pi a^2} \leq \mathcal{P}^{\text{cap}} \leq -\frac{\Phi z_c F_0}{\pi a^2}$. (This relation between \mathcal{P}^{cap} and contact tensile strength F_0 is sometimes referred to as the Rumpf formula, especially in the context of a prediction of rupture conditions [13, 19, 50]). Dividing by σ_{22} , one obtains:

$$-\frac{\Phi z}{\pi P^*} \leq \frac{\mathcal{P}^{\text{cap}}}{\sigma_{22}} \leq -\frac{\Phi z_c}{\pi P^*}. \quad (16)$$

Ignoring the small difference between \mathcal{P} and σ_{22} , (16) provides an estimate of coefficient β defined in (15). Thus the value of β for reduced pressure $P^* = 0.436$ is predicted between 1.9 and 2.3 (and for $P^* = 0.1$, it should reach about 8). Thus, quite unsurprisingly, the (negative) relative contribution of capillary forces to normal stress is of order $(1/P^*) \propto F_0/P$, with a coefficient that may be deduced from Φ and coordination numbers, according to (16).

It is tempting to invoke a classical concept in geomechanics, that of *effective pressure*, to describe the effect of capillary forces on the shear resistance of the material: the attractive forces create larger repulsive elastic reactions in the contact, corresponding to an effective pressure equal to $(1 + \beta)\mathcal{P}$. Furthermore, the local Coulomb condition in the contacts is to be written with those enhanced normal repulsive forces. Capillary forces also contribute to shear stress, but, as apparent in Fig. 17, in comparison to their influence on normal stresses, this is a small effect, and one may ignore it in a first approach.

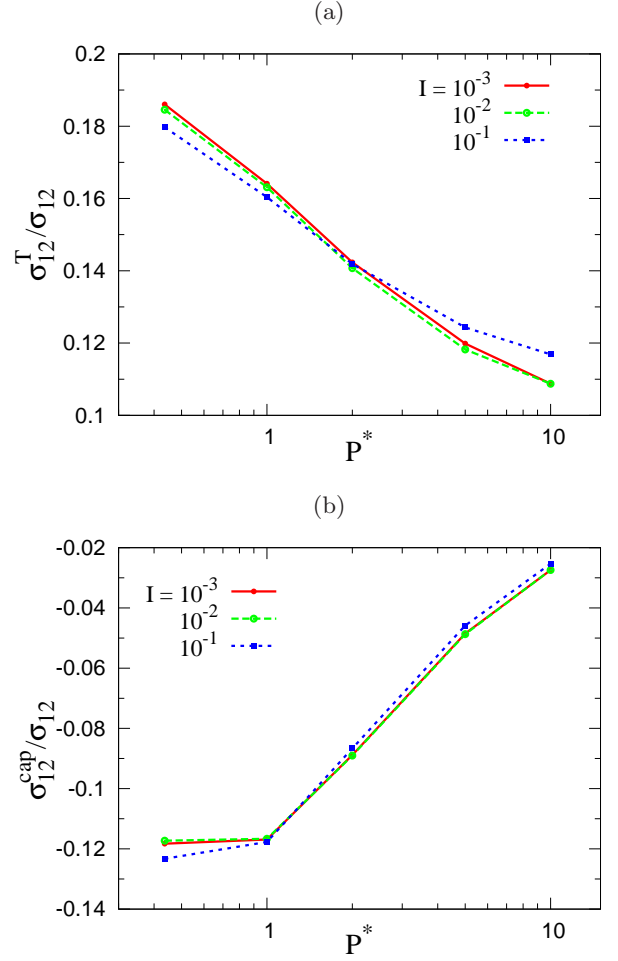


FIG. 17: (Color online) Contributions of tangential (a) and capillary (b) interactions to total shear stress σ_{12} .

One assumes then that the shear behavior of the material is identical to that of a dry material under such effective normal stress σ_{22}^{eff} . This approach leads to the following prediction for the P^* -dependent quasistatic friction coefficient μ_0^* :

$$\mu_0^* = (1 + \beta)\mu_0^\infty, \quad (17)$$

in which μ_0^∞ denotes the quasistatic internal friction coefficient for dry grains, $P^* = \infty$. Remarkably, if we further assume, as suggested by (16), that β is roughly proportional to $1/P^*$, $\beta \simeq b/P^*$, we obtain a Mohr-Coulomb relation, Eq. 1, for the stresses in the critical state: with the same value of internal friction as in the dry case, $\mu_1^* = \mu_0^\infty$ and a macroscopic cohesion given by

$$c = \frac{b\mu_0^\infty F_0}{a^2}. \quad (18)$$

Fig. 18 is a plot of σ_{12} versus σ_{22} – the yield locus – in which the predictions of relation 17, both with the measured coefficient β (Fig. 16), and with the one predicted as $(z + z_c)\Phi/(2\pi P^*)$ from (16), are confronted with the

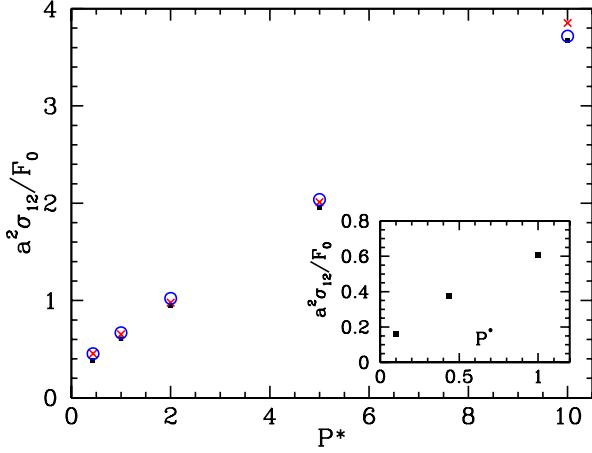


FIG. 18: (Color online) σ_{12} versus σ_{22} , in quasistatic flow, in units of F_0/a^2 . Square dots: numerical results (error bars are smaller); red crosses: predictions of (17), with exact coefficient β ; blue circles: same with estimated β . Insert: detail of numerical data for small P^* , including additional point at $P^* = 0.1$.

numerical results. The admittedly crude prediction of relation (17) appears surprisingly close to the numerical results on this plot. The relative error in the prediction for stress ratio μ_0^* , with the measured value of β , is actually about 5% at $P^* = 10$, increasing to 20% at $P^* = 0.436$, and the value of μ_0^* for $P^* = 0.1$ ($\simeq 1.6$) from the measurements at $I = 0.01$ is largely overestimated, at 2.7.

One may directly test for the validity of a Mohr-Coulomb relation to the data by fitting a linear form for the data of Fig. 18. Given the error bars (which are small and do not appear on the graph), an attempted straight line fit through all 5 data points with $P^* \geq 0.43$ in Fig. 18 is unambiguously rejected by the standard likelihood criterion. A linear fit is (barely) acceptable upon ignoring the value $P^* = 0.436$, yielding $\mu_1^* = 0.340 \pm 0.001$ and $a^2 c / F_0 = 0.267 \pm 0.005$ for the Mohr-Coulomb parameters. From (17) the predicted apparent macroscopic cohesion is above $0.3F_0/a^2$, and varies according to which data are used to identify b in (18). The result $\mu^* \simeq 1.6$ for $P^* = 0.1$ (corresponding to $a^2 \sigma_{12} / F_0 = \mu^* P^* = 0.16$) is thus in contradiction with the Mohr-Coulomb model, which becomes increasingly inadequate for smaller P^* , as apparent in the insert in Fig. 18.

The performance of the simple effective pressure prediction for the P^* dependence of μ_0^* is better visualized in Fig. 19, which, unlike Fig. 18, is not sensitive to stress scale. The global increase of μ_0^* is predicted, yet overestimated for the smallest P^* values.

One aspect that is not captured by this approach is the dependence of μ_0^* on meniscus volume (Tab. IV): the variations of coefficient β (from 1.8 to about 2.2 as V increases from $10^{-6}a^3$ to $5.10^{-3}a^3$) are insufficient to account for the increase of the friction coefficient.

There are quite a few reasons for the effective pres-

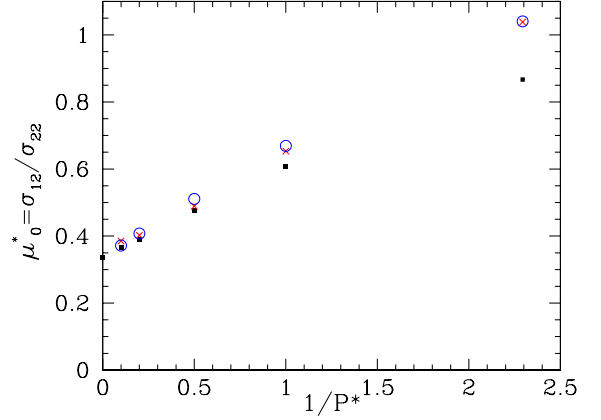


FIG. 19: (Color online) Apparent quasistatic friction coefficient μ_0^* versus $1/P^*$ – showing the value of μ_0^∞ for $1/P^* = 0$. Measurements and predictions of (17), with exact and estimated coefficient β , same symbols as in Fig. 18.

sure approach to fail: while the mechanical properties are supposed to be the same once stresses are corrected, the density of the material, for one thing, is different in the dry and the wet case (with Φ varying between 0.525 and 0.595 as P^* grows from 0.46 to infinity); capillary forces also contribute to shear stress, the force network is bound to be different, etc. Nevertheless, although admittedly crude, the prediction based on (17) proves apt to capture the trend of the change of μ_0^* with P^* , although it overestimates its growth at small P^* . As to the Mohr-Coulomb representation of yield stresses, it might be used as an approximation for $P^* \geq 1$, but the observations clearly preclude the definition of unique values of macroscopic cohesion and friction coefficient according to (1) for smaller pressures.

Ref. [50] reports on a laboratory study of quasistatic yield loci (σ_{12} versus σ_{22} at the onset of plastic yielding and flow) of various kinds of wet granular assemblies in the pendular regime, including glass beads, which offer a suitable experimental comparison to our results. It proposes (under the name 'shift theory'), exactly the same effective pressure approach as the one we have attempted here, and concludes that it provides a good approximation, by which the yield condition of wet materials is deduced from the one of the dry grains. Interestingly, the investigated P^* values in this study range from about 0.2 to ~ 2.5 , and the yield locus is slightly concave, as in our numerical results. Measured values of μ^* are similar to our results (with e.g., $\mu^* \simeq 0.7$ for $P^* = 1$), and little change is obtained by increasing saturation by a factor of 3. Some possible differences between those experiments and our simulations could result from the different state of the material (the experiments are not necessarily carried out in steady state quasistatic shear flow, and could depend on the initial assembling process), and from the value of the wetting angle. However, quite a satisfactory semi-quantitative agreement is obtained.

In the following sections, for a better assessment of the rheophysical effect of attractive capillary forces, microscopic and microstructural aspects of force networks are investigated in greater detail.

VI. FORCE DISTRIBUTION

The distribution of intergranular force values in a granular material in equilibrium [31, 51–54], or in inertial flow [1, 5] has received a lot of attention in the recent literature. While the probability distribution function of force values in cohesionless systems tends to decrease exponentially, on a scale given by the average $\langle F_N \rangle$, in cohesive granular assemblies, characterized by the contact tensile strength F_0 , the equilibrium force distribution evolves, as P^* decreases to low values, towards a roughly symmetric distribution about zero, with values of both signs of order $F_0 \gg \langle F_N \rangle$ [13, 15]. As compared to the two-dimensional results of Refs. [13, 15], the present 3D numerical study of wet spherical grain assemblies does not investigate very small P^* states, but involves longer-ranged distant interactions. The positive force wing of the p.d.f. of normal forces near the quasistatic limit is shown in Fig. 20, showing the gradual departure from

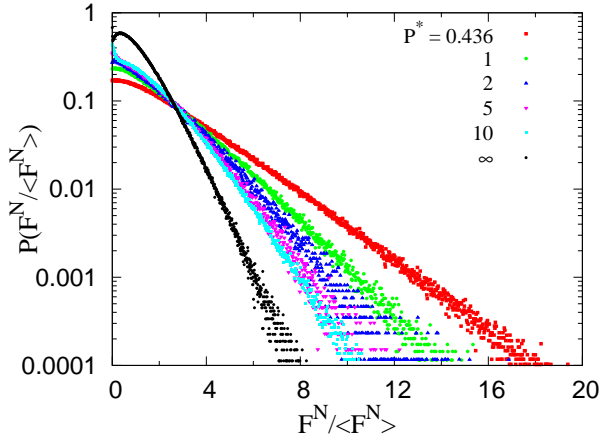


FIG. 20: (Color online) Distributions of normal forces normalized by average normal force $\langle F_N \rangle$ for $I = 10^{-3}$ and different values of P^* .

the cohesionless distribution shape, and the transition to a cohesion-dominated force network with values of order F_0 , ratio $\langle F_N \rangle / F_0$ being approximately proportional to P^* as discussed in Sec. VB.

At low reduced pressure, as for $P^* = 0.436$, it is more appropriate to normalize the distribution by F_0 , as in Fig. 21. This plot shows the influence of inertia parameter I , which is, for large positive values, qualitatively similar to the one observed with dry grains: the distribution widens, large forces being associated with collisions between grains or groups of grains. Another effect of increasing the inertial number is, as expected from the

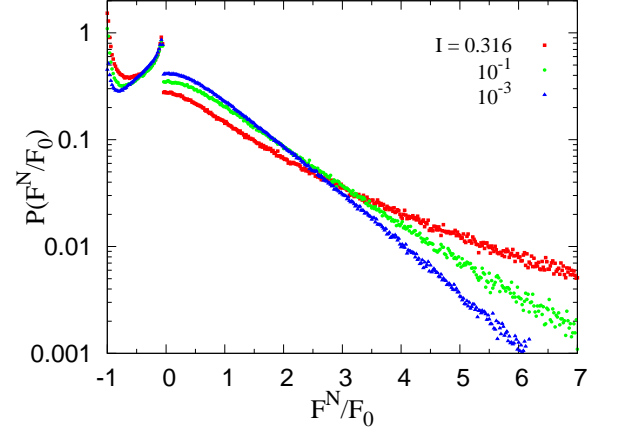


FIG. 21: (Color online) Distributions of normal forces for $P^* = 0.436$ and different values of I , normalized with F_0 .

results of Fig. 12, a depletion of the population of contacts, compensated by a greater number of distant grains joined by a meniscus. To understand better the distribution shape for negative values, Fig. 22 distinguishes the distributions of contact and distant (attractive) forces. Contact force distributions exhibit a maximum in zero, with negative values becoming more frequent as I increases. The larger value of the pdf near $-F_0$ signals then the opening of more contacts. The distant interac-

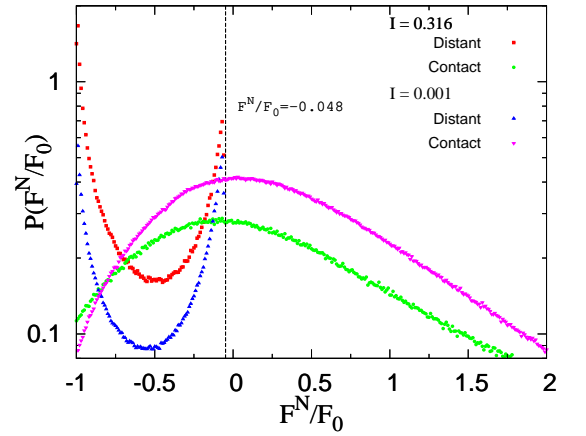


FIG. 22: (Color online) Contributions of contact and distant interactions to p.d.f. of normal forces for $P^* = 0.436$, and two values of I , $I = 0.316$ and $I = 0.001$. The vertical dashed line corresponds to force at rupture distance, $F^{\text{Cap}}(D_0)$.

tions are responsible for the non-monotonic part of the pdf. On the one hand, the sharp maximum near $-F_0$ signals a large population of grain pairs at close distance, in agreement with the fast increase of $z(h)$ at small h visible in Fig. 14. On the other hand, the increase near the minimum attractive force at rupture distance D_0 merely

reflects the slow variation of function $F^{\text{Cap}}(h)$ (Fig. 2).

The "effective pressure" concept relies on the assumption that the effect of attractive capillary forces are similar to that of a larger applied isotropic stress. One way to test such an idea at the microscopic scale is to compare the distributions of normal elastic forces: if normalized by the average elastic force, related to the effective pressure, those should be independent on P^* and similar to the force distribution in a cohesionless system. Fig. 23

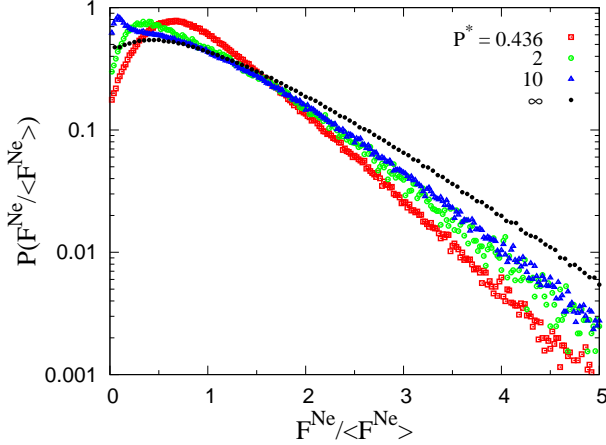


FIG. 23: (Color online) Distributions of normal elastic forces at small I for different P^* values.

compares the distributions of elastic normal forces, normalized by their mean value, for small I and different values of P^* . Those distributions are roughly similar, but show, as expected, notable discrepancies for values of order F_0 . The decay for large values is faster in cohesive systems, reflecting a difference in force networks.

VII. AGGLOMERATION

The aggregation of cohesive grains is observed and reported in many numerical and experimental studies, and is exploited in industrial processes [16, 55]. It was directly observed in flow of cohesive granular assemblies, both in numerical model materials [16], and in experiments with wet powders [56]. A numerical study of steady state chute flow [57] reports an increase of the number of long-lasting contacts in the presence of cohesive forces. Weber *et al.* in [58], carried out a detailed study of the effect of capillary forces on agglomerate duration and size. The agglomeration phenomena in steady shear flow is studied here, first, by measuring contact ages and meniscus ages, depending on state parameters. Then, the age-dependent size of clusters is measured, depending on P^* and I . These clustering properties are related to the material rheology.

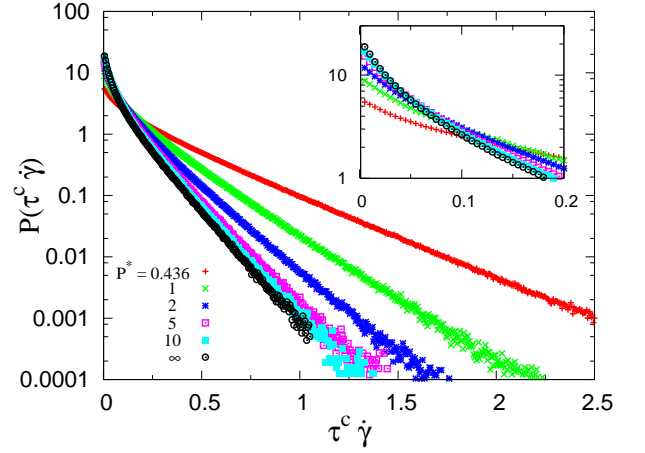


FIG. 24: (Color online) Distribution of the age of contacts for different values of P^* and $I = 0.1$. Inset shows the same graph in a shorter range of $\tau^c \dot{\gamma}$.

A. Age of contacts and of distant interactions

The distribution of the age τ^c of contacts for $I = 10^{-1}$ and different values of P^* is shown in Fig. 24. $P(\tau^c \dot{\gamma})$ is the probability distribution of contact ages τ^c , expressed as a strain $\tau^c \dot{\gamma}$. The decrease of $P(\tau^c \dot{\gamma})$ is slower for smaller P^* , showing that for the stronger cohesive forces the contacts survive over larger strain intervals [57, 58]. For large enough strains, $\tau^c \dot{\gamma} > 0.5$, these probability density functions decay with an exponential form, $P(\tau^c \dot{\gamma}) \propto e^{-\tau^c/\tau_0}$. Values of decay times τ_0 , given in Tab. VI, increase as we decrease P^* . Average contact ages, τ_{avg}^c , also provided in the table, show the same behavior (τ_{avg}^c is smaller than τ_0^c because the distribution is not exponential for short times – see inset on the figure). Fig. 25 shows the evolution of the pdf with I for two different values of P^* , revealing, as expected, that contact ages (in units of $1/\dot{\gamma}$) decrease in faster flows. For $I \leq 10^{-2}$, curves appear to coincide, showing nearly quasistatic behavior. The probability distribution function of the age of interactions $P(\tau^i \dot{\gamma})$ (i.e., the age of liquid bridges) is also shown in Fig. 26 for different values of P^* . Liquid bridges survive for quite large strain intervals, reaching several units with a probability of order 0.1, which increase as P^* decreases. Initially, most liquid bridges survive at least for strains of order 0.1. Beyond unit strain curves might be fitted by an exponential function too, defining a decay time τ_0^i . P^* -dependent values of τ_0^i and of the average meniscus age τ_{avg}^i are listed in Tab. VI. Remarkably, the curves do not present any notable difference for different values of I : the pairs may lose their contacts in faster flows, but they are still bonded with liquid bridges. The age of contacts and of distant interactions thus reveal the formation of aggregates in the presence of capillary forces. These clusters are transported by the flow for some distance before they

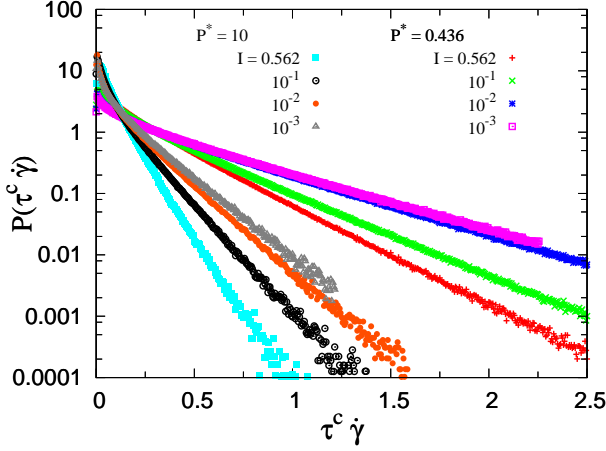


FIG. 25: (Color online) Distribution of the age of contacts for different values of I and two different values of P^* .

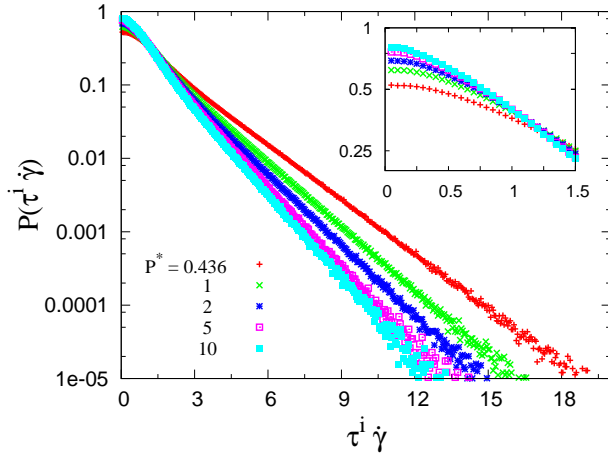


FIG. 26: (Color online) Distribution of the age of menisci for different values of P^* (same for all values of I). Inset shows the same graph in a shorter range of $\tau^i \dot{\gamma}$.

are broken or restructured. They may survive for strain intervals of a few units.

B. Clusters

Clusters are defined as sets of grains connected by liquid bonds for a minimum time, τ^{cl} , and the clustering tendency might be appreciated on recording the τ^{cl} -dependent mass-averages, cluster size:

$$\langle S^{\text{cl}} \rangle_{\text{m}} = \frac{\sum_i S_i^2}{\sum_i S_i}, \quad (19)$$

in which the summations run over all clusters i containing S_i grains. The results for $\langle S^{\text{cl}} \rangle_{\text{m}}$ for the different values of P^* and I are shown in Fig. 27, as functions of $\tau^{\text{cl}} \dot{\gamma}$. For small values of $\tau^{\text{cl}} \dot{\gamma}$ almost all particles are gathered

P^*	$\dot{\gamma} \tau_0^{\text{c}}$	$\tau^{\text{c}} \dot{\gamma}_{\text{avg}}$	$\dot{\gamma} \tau_0^{\text{i}}$	$\tau^{\text{i}} \dot{\gamma}_{\text{avg}}$
0.436	0.306	0.258	1.704	1.609
1	0.180	0.154	1.437	1.325
2	0.153	0.111	1.295	1.187
5	0.128	0.087	1.164	1.072
10	0.120	0.080	1.102	1.021
∞	0.118	0.074	—	—

TABLE VI: (Color online) Decay time of age distribution function for contacts, τ_0^{c} , and for all interactions, τ_0^{i} , obtained by an exponential fit to the data of Fig. 24 and of Fig. 26); average contact age $\tau_{\text{avg}}^{\text{c}}$ and interaction age $\tau_{\text{avg}}^{\text{i}}$, for different values of P^* and $I = 0.1$. All four times are normalized by shearing time $1/\dot{\gamma}$.

in a single cluster, which results in $\langle S^{\text{cl}} \rangle_{\text{m}} \simeq 4000$. For large values of $\tau^{\text{cl}} \dot{\gamma}$ most particles are isolated or part of a very small cluster, and so a value of order 1 for $\langle S^{\text{cl}} \rangle_{\text{m}}$ is expected. In the midrange of $\tau^{\text{cl}} \dot{\gamma}$, $\langle S^{\text{cl}} \rangle_{\text{m}}$ increase for decreasing P^* or I , i.e. for stronger capillary forces or slower flows.

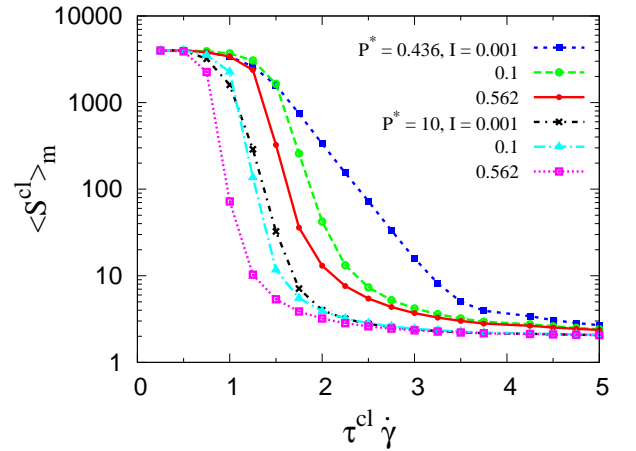


FIG. 27: (Color online) Mass-averaged cluster size, $\langle S^{\text{cl}} \rangle_{\text{m}}$, versus cluster age $\tau^{\text{cl}} \dot{\gamma}$ for different values of P^* and I .

VIII. FABRIC ANISOTROPY

The capacity of granular assemblies to form anisotropic force networks is the only origin of shear strength with frictionless grains [5, 54, 59], and is known to play a central role in the shear strength of frictional grains as well. To understand how contact and distant capillary forces contribute to the shear stress, it is instructive to study the distribution of contact orientations (normal vectors \mathbf{n} on the unit sphere Σ), $E(\mathbf{n})$, which, to lowest order, is characterized by the fabric tensor:

$$F_{\alpha\beta} = \langle n_\alpha n_\beta \rangle = \int_{\Sigma} E(\mathbf{n}) n_\alpha n_\beta d^2 \mathbf{n} \quad (20)$$

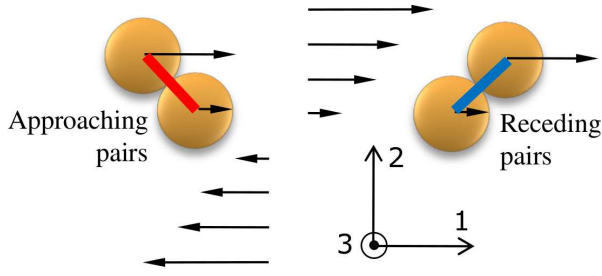


FIG. 28: (Color online) Sketch of approaching (φ near $3\pi/4$) and receding pairs (φ near $\pi/4$) in macroscopic shear flow.

The connection between fabric and normal force contribution to stresses, or $\underline{\underline{\sigma}}^N$, is quite direct, as one has:

$$\sigma_{\alpha\beta}^N = \frac{z\Phi}{3\pi a^2} \int_{\Sigma} E(\mathbf{n}) \langle F^N \rangle_{\mathbf{n}} n_{\alpha} n_{\beta} d^2 \mathbf{n}, \quad (21)$$

an integral over the unit sphere in which $\langle F^N \rangle_{\mathbf{n}}$ denotes the average normal force carried by the pairs with orientation \mathbf{n} . As reported in Sec. V C, the contribution of the normal forces, $\underline{\underline{\sigma}}^N$, amounts to more than 80% of the shear stress. The contribution of fabric parameters F_{12} to shear stress σ_{12} might be visualized in Fig. 28. On average, if pairs are preferentially oriented with the normal vector within a compression quadrant in the shear flow, then $F_{12} < 0$ will tend to increase the absolute value of σ_{12} if forces are positive, and to decrease it they are negative. On the other hand, negative forces will increase the absolute value of σ_{12} if preferentially oriented in the extension quadrants in the shear flow.

The evolutions of fabric parameters F_{12}^c and F_{12}^d , pertaining respectively to contact and distant normal forces, versus I , for different P^* values, are displayed in Fig. 29. F_{12}^c is negative, signaling the contribution of normal contact forces to shear strength (as $\langle F^{N,c} \rangle$ is the dominant contribution to $\langle F^N \rangle$, related to \mathcal{P} by (12)). The largest value, and the largest variation of F_{12}^c with I , is obtained in the dry system ($P^* = \infty$). The decrease of this anisotropy parameter for smaller P^* may be understood in reference to the clustering phenomena and to the larger duration of contacts evidenced in Sec. VII. Longer-lived contacts rotate in the shear flow, and are less favorably oriented in the compression quadrant. Shear flow carries agglomerates for some distance before they break and thus their random tumbling motion increases the isotropy of the contact orientations. In faster flows (for larger I), while contacts tend to open in cohesionless materials, enhancing fabric anisotropy, cohesive contacts can resist flow agitation and inertial effects better, whence a smaller I influence; if they open, they transform into attractive distant interactions, and the anisotropy of distant interactions also decreases. Those distant capillary forces are characterized by a comparatively large anisotropy, about three times as large as $|F_{12}^c|$. As F_{12}^d is positive, those distant attractive forces contribute to increase the internal friction coefficient. Unlike $|F_{12}^c|$, F_{12}^d increases for smaller

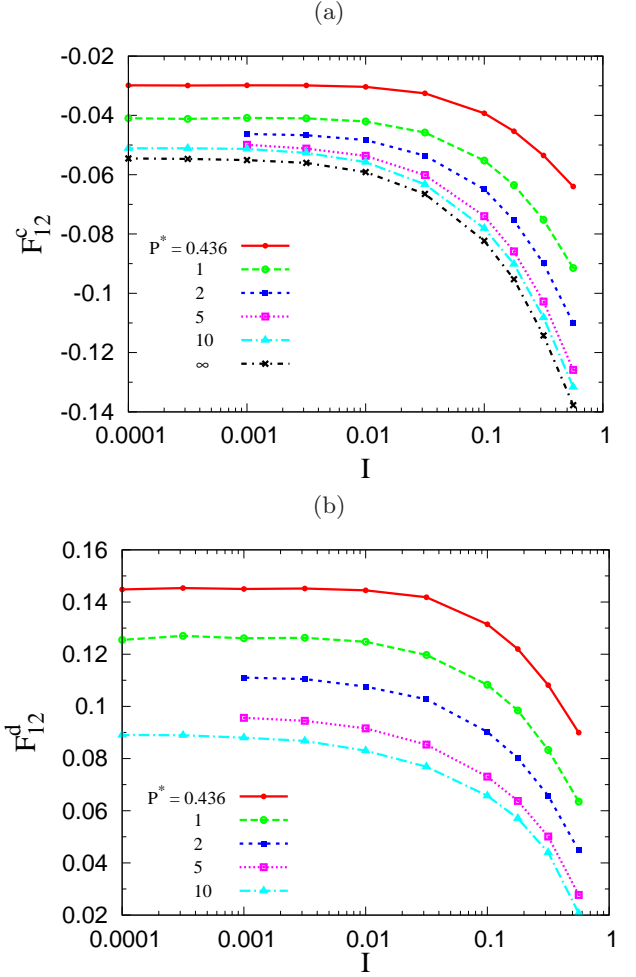


FIG. 29: (Color online) Fabric parameter F_{12} for contacting pairs (a) and distant interactions (b) versus I for different P^* .

P^* values, which corresponds to the growing contribution of distant interactions to shear strength shown in Fig. 15. The different rule of meniscus formation (at contact) and breakage (at distance D_0) explains, in part, this large fabric anisotropy of attractive forces: approaching particles are not attracted to each other, whence a small number of distant interacting pairs in the compressive quadrant, with a negative contribution to F_{12} ; as particles get separated, receding pairs are still attracted to each other, whence a positive contribution to F_{12} from the extension quadrant. In the model without meniscus hysteresis, assuming capillary attraction to appear as soon as grains approach within distance D_0 , F_{12}^d strongly decreases, from 0.14 to about 0.07 at $P^* = 0.436$ and small I .

IX. SUMMARY AND DISCUSSION

The rheological properties of unsaturated granular materials, in which a small amount of wetting liquid, form-

ing liquid bridges and transmitting attractive capillary forces between particles, generalize, in many respects, previous observations on cohesive granular materials, with macroscopic properties exhibiting similar dependences on I and P^* . Thus, compared to dry materials, the apparent internal friction coefficient $\mu^* = \sigma_{12}/\sigma_{22}$ is enhanced (from 0.33 to more than 1 in the explored range $P^* \geq 0.1$); looser structures are stabilized, even in the quasistatic limit ($\Phi \simeq 0.52$, for $P^* = 0.436$ is below all packing densities with cohesionless grains), even though contact coordination numbers, due to the absence of rattlers, may be larger. Our results describe those effects in quantitative form, in the range $P^* \geq 0.1$ and $10^{-4} \leq I \leq 0.56$, and specify the dependence on various features of the model. We only predict quite a small dependence of the rheology on saturation within the pendular range (up to 5–10%), in agreement with experimental observations [19, 50].

More accurate models of the capillary force dependence on intergranular distance, or of the distribution of liquid between menisci with varying volumes, would hardly change the results. Interestingly, though, some variants of the model, although arguably not realistic, have notably different rheological properties. Thus, reducing the meniscus volume to very small values would have quite a notable effect on internal friction and density – but, in practice, menisci are unlikely to form with such small liquid contents. Assuming menisci form as soon as grains approach to their maximum extension distance (range of capillary force) would also strongly affect macroscopic properties.

Shear localization systematically affects shear flows at low P^* , and we could not measure the constitutive behavior at $P^* = 0.1$ except for some intermediate I values of order 0.01. Localized states are characterized by velocity profiles with gradients concentrated within narrow bands, where the solid fraction is well below its bulk value. The band thickness lies in the range of 5 to 10 grain diameters a at small I , but might be as small as about $1.5a$ in faster flows ($I \geq 0.1$).

We also record normal stress differences, which are larger than for dry grains, and tend to grow with decreasing P^* . The second normal stress difference, in particular, reaches 20% of the imposed normal stress σ_{22} in the quasistatic limit for small P^* .

The effective pressure approach to the yield criterion of wet grains ignores such sophistications, as well as density or microstructural changes due to capillary forces. It assumes critical states to be in correspondence for different values of P^* , as though the introduction of capillary forces, pushing grains against their neighbors, were equivalent to the application of a larger confining pressure. Such a crude approach is in fact surprising successful as a rough approximation, to predict the increase of μ^* for decreasing, but not too small, P^* , say $P^* \geq 1$ (below $P^* = 1$, the increase of μ^* is overestimated). The

effective pressure might be evaluated upon adding, to the applied pressure, the capillary contribution to the average stress. This contribution might itself be estimated from density and coordination numbers, which leads to a Mohr-Coulomb form (1) for the variation of tensile strength $\mu_0^* \sigma_{22}$ with normal stress σ_{22} (if Φ and z do not vary too much). Such a form of the critical state plasticity criterion proves however inadequate to describe the whole range of reduced pressures P^* : data are incompatible with a P^* -independent macroscopic cohesion c .

Another remarkable feature of the measured rheology, compared to dry granular materials, is the slow variation with I of both macroscopic rheological parameters such as μ^* , and microscopic data such as coordination numbers. Although we could not observe a direct correlation, a slower increase of function $\mu^*(I)$ could signal a shear banding tendency.

Many of those rheophysical features are explained by, or, at least, related to, the strong clustering tendency emerging as attractive forces gradually become the dominant ones, upon decreasing P^* . As contacts are stabilized by attractive forces, they do not so easily open as the network is being sheared. When they do (which happens preferentially in the extension direction within the average shear flow, whence a fabric anisotropy of distant interactions contributing to shear strength), the network of grains bonded by liquid bridges might still be connected, forming enduring connected clusters. The survival of such clusters over quite notable strain intervals (reaching several unities with sizable probability) should limit the dilating tendency of faster flows. It also maintains a network in which the capillary forces act in closer similarity to an effective pressure.

Admittedly, this is still a descriptive rheophysical scenario. More quantitative studies should be carried out to better characterize the deformation mechanisms of the grain clusters. The shear banding phenomenon certainly deserves detailed investigations, in which sample size and shape effects should be systematically assessed, and partly localized velocity and density fields analyzed and related to a stability analysis.

Although we argued that our measurements agree with the limited available experimental results, more laboratory data should certainly be used, with, if possible both information on rheology and on micromorphology and liquid distribution.

Finally, a complete numerical model should eventually be designed, capable of dealing with saturations exceeding the limited pendular range, and of describing the liquid motion. One should thus be equipped to model the mixing process as well as the rheology of the homogeneous mixture of the grains and the liquid. The Lattice Boltzmann method for a diphasic interstitial fluid medium, coupled to a DEM description of grain motion, is a promising perspective [60, 61].

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